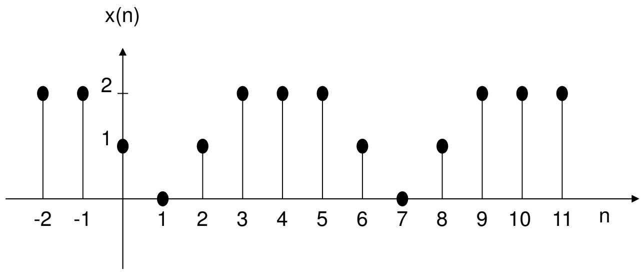
EN5101 Digital Control Systems
z-Transforms

Prof. Rohan Munasinghe
Dept of Electronic and Telecommunication Engineering
University of Moratuwa

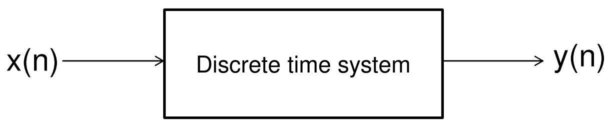
Contents

- Forward z-transform
- Inverse z transform methods
  - Power series, Recursive, Partial Fraction expansion
- z-transform Applications
- Applications of z-transform
  - Pole-zero description of discrete-time systems
  - Stability consideration
  - Difference Equations

Discrete-time (sampled) Signal



- A discrete time system is essentially a mathematical algorithm that takes an input sequence, x(n), and produces an output sequence, y(n).
- Example of discrete time systems are digital controllers, digital spectrum analyzers, and digital filters.



Analysis of Sampled Data Systems

$u(k)$, $u(k-1)$, $u(k-2)$, $\vdots$, $u(k-m)$ → $\boxed{P}$ → $y(k)$, $y(k-1)$, $y(k-2)$, $\vdots$, $y(k-n)$

input sequence — output sequence

a causal LTI system can be generally described by

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \dots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \dots \dots b_m u(k-m)$$

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + \dots - a_n y(k-n) + b_0 u(k) + b_1 u(k-1) + \dots \dots + b_m u(k-m)$$

where $n \geq m$ (there cannot be an input without causing an output).

* State space model of time-shifted invariant "

Problem with the Laplace in discrete systems

Digital or Sampled systems (solution of difference eq.) → signals contain time delay

e^{-Ts} ← s in exponent → not rational

e^{Ts} in Laplace domain

↓

difficult to use Laplace transforms

↓

let's introduce Z-transforms

$z = e^{Ts}$ $\quad \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt$

$Z[f(k)] = \sum f(k) z^{-k}$ delay operator

$f(t)$ → sampler → $f^*(t) \xrightarrow{\mathcal{L}} F^*(s) \xrightarrow[z=e^{Ts}]{} F(z)$

steps to obtain z-transform: $f(kT)\delta(t-kT)$ $\quad \mathcal{L}[\;] = e^{-skT} \mathcal{L}[f(k)]$

Laplace (Continuous) to z (Discrete)

1. Sample $f(t) \longrightarrow f^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t-kT)$, $k=0,1,2,\ldots$ $= e^{-skT} f(kT)$
2. $F^*(s) = \mathcal{L}[f^*(t)] = \sum_{k=0}^{\infty} f(kT) e^{-skT}$ → not a rational f^n of s
3. $F(z) = \sum_{k=0}^{\infty} f(kT) z^{-k}$; $z = e^{Ts}$ → is rational f^n of z

The z-transform

- The z-transform of a sequence, x(n), which is valid for all n, is defined as power series

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where X(z) = Z{x(n)} and z=**re**$^{j\omega}$ is a complex variable

- In causal systems x(n) = 0 for n < 0, x(n) may be nonzero value only in the interval 0< n< ∞, so one sided z-transform can be written as follow.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

- Clearly, the z-transform is a power series with an infinite number of terms and so may not converge for all values of z.

Region of convergence (ROC) of Z- transform

- The region of convergence (ROC) of X(z) is the set of all values of z for which X(z) attains a finite value.
- The region where the z-transform converges is known as the *region of convergence* (ROC) and in this region, the values of X(z) are finite.
- Thus, any time we cite a z-transform we should also indicate its ROC.

Poles & Zeros of X(z)

- Values of z for which X(z) attains ∞ are referred to as poles of X(z).
- Values of z for which X(z) attains 0 are referred to as the zeroes of X(z).

For example

$$X(z)=\frac{3z^2-13z-10}{z^2+2z-8}$$

Find the poles and zeroes of X(z).

Solution:

$$X(z)=\frac{3z^2-13z-10}{z^2+2z-8}=\frac{(3z+2)(z-5)}{(z+4)(z-2)}$$

- Poles

(z + 4) (z - 2) = 0 ⇒ z + 4 = 0 and z – 2 = 0

z = -4 and z = 2

- Zeroes

(3z + 2) (z - 5) = 0 ⇒ 3z + 2 = 0 and z - 5 = 0

z = -2/3 and z = 5

Example

Determine the z-transform of the following finite-duration signals.

(a) $x_1(n)$ = [1,2,5,7,0,1]
↑

(b) $x_2(n)$ = [1,2,5,7,0,1]
↑

(c) $x_3(n)$ = [0,0,1,2,5,7,0,1]
↑

(d) $x_4(n)$ = [2,4,5,7,0,1]
↑

(e) $x_5(n) = \delta(n)$

(f) $x_6(n) = \delta(n - k)$, k >0

(g) $x_7(n) = \delta(n + k)$, k >0

Solution

$x_1(n)$ = [1,2,5,7,0,1]

$x_1(n)$ is finite length sequence. It is causal case and having n = 0 to 5. Therefore,

$x_1(0)=1$, $x_1(1)=2$, $x_1(2)=5$, $x_1(3)=7$, $x_1(4)=0$, $x_1(5)=1$

$$\begin{aligned}X_1(z)&=\sum_{n=0}^{5}x_1(n)z^{-n}\\&=x_1(0)z^{-0}+x_1(1)z^{-1}+x_1(2)z^{-2}+x_1(3)z^{-3}+x_1(4)z^{-4}+x_1(5)z^{-5}\\&=1+2z^{-1}+5z^{-2}+7z^{-3}+0z^{-4}+1z^{-5}\\&=1+2z^{-1}+5z^{-2}+7z^{-3}+z^{-5}\end{aligned}$$

The ROC is entire z-plane except at z = 0.

$X_2(n) = [1,2, 5,7,0,1]$

$X_2(n)$ is finite length sequence and double sided (non-causal case) having n=-2 to 3. Therefore,

$x_2(-2)=1$, $x_2(-1)=2$, $x_2(0)=5$, $x_2(1)=7$, $x_2(2)=0$, $x_2(3)=1$

$$
\begin{aligned}
X_2(z) &= \sum_{n=-2}^{3} x_2(n) z^{-n} \\
&= x_2(-2)z^{-(-2)} + x_2(-1)z^{-(-1)} + x_2(0)z^{-0} + x_2(1)z^{-1} + x_2(2)z^{-2} + x_2(3)z^{-3} \\
&= 1z^2 + 2z^1 + 5z^0 + 7z^{-1} + 0z^{-2} + 1z^{-3} \\
&= z^2 + 2z + 5 + 7z^{-1} + z^{-3}
\end{aligned}
$$

The ROC is entire z-plane except at z = 0 and z = ∞.

(e) $x_5(n) = \delta(n)$

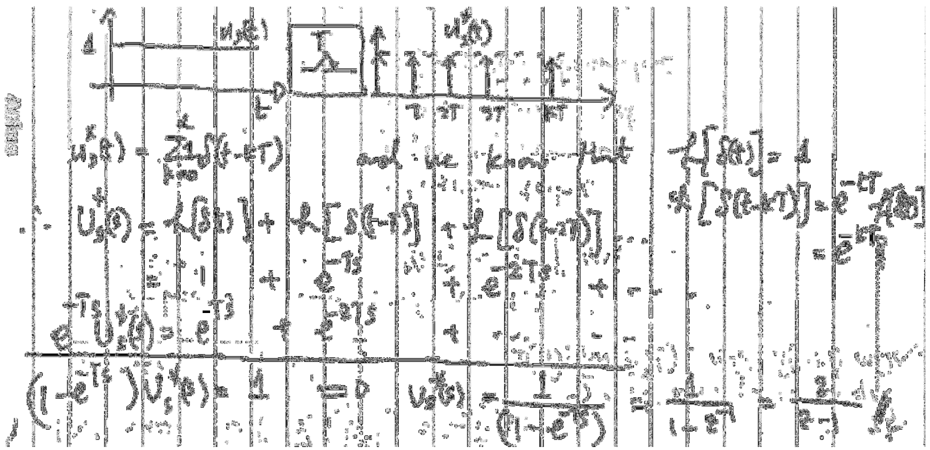
By definition,

$$\delta(n) = \begin{cases} 1 & for \quad n = 0 \\ 0 & else \end{cases}$$

By using general formula,

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = \delta(0) z^{-0} \\
&= 1 \times 1 = 1
\end{aligned}
$$

z Transform of a sampled unit step



$if\ |e^{-Ts}| = |z^{-1}| \leq 1$

Example

Determine the z-transform of the signal

$x(n) = (1/2)^n u(n).$

Solution:

$$u(n) = \begin{cases} 1 & for \quad n > 0 \\ 0 & for \quad n < 0 \end{cases}$$

The signal x(n) consists of an infinite number of non-zero samples for n> 0.

$x(n) = [1, (1/2), (1/2)^2, (1/2)^3, \ldots\ldots\ldots\ldots, (1/2)^n, \ldots.]$

The z-transform of the x(n) is the infinite power series

$X(z) = 1 + ½ z^{-1} + (1/2)^2 z^{-2} + \ldots\ldots(1/2)^n z^{-n} + \ldots.$

$$X(z)=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}z^{-n}=\sum_{n=0}^{\infty}(\frac{1}{2}z^{-1})^{n}$$

- This is an infinite geometric series, we recall that

$$1+a+a^{2}+a^{3}+.........=\frac{1}{1-a}\ if\ |a|<1$$

where 'a' is the common ratio of the series. Consequently,

$$X(z)=\frac{1}{1-(1/2)z^{-1}}\quad if\quad \left|(1/2)z^{-1}\right|<1$$

$$\text{ROC}\ :\ |z|>1/2$$

Example

Find the z-transform and the region of convergence for each of the discrete-time sequence

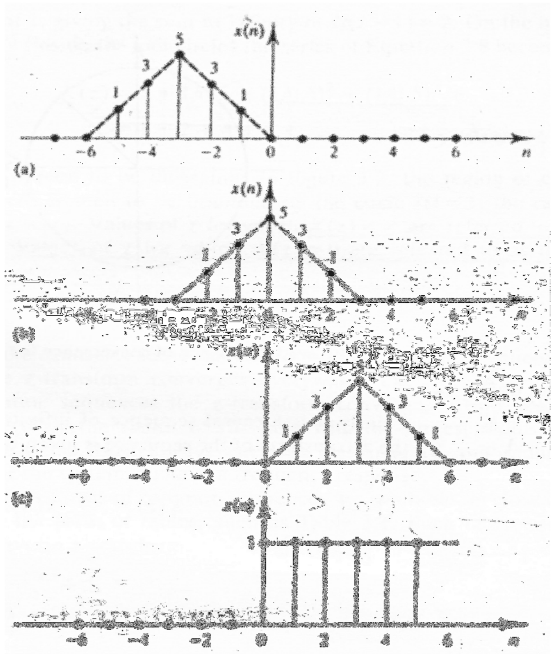


Fig 1 sample sequences

(a) The z-transform is given by

$$X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}=\sum_{n=-7}^{6}x(n)z^{-n}=\sum_{n=-6}^{0}x(n)z^{-n}$$

$$X_1(z)=(-6)z^{-(-6)}+x(-5)z^{-(-5)}+x(-4)z^{-(-4)}+x(-3)z^{-(-3)}$$

$$...............+x(-2)z^{-(-2)}+x(-1)z^{-(-1)}+x(0)z^{0}$$

$$X_1(z)=0z^{-(-6)}+1z^{-(-5)}+3z^{-(-4)}+5z^{-(-3)}+3z^{-(-2)}+1z^{-(-1)}+0z^{0}$$

$$X_1(z)=z^{5}+3z^{4}+5z^{3}+3z^{2}+z^{1}$$

- It is readily verified that the value of X(z) becomes infinite when z = ∞. Thus the ROC is everywhere in the z- plane except at z = ∞.

(b) Again, the sequence in figure 1(b) is not causal. It is of a finite duration, and double sided.

$$X_2(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}=\sum_{n=-4}^{5}x(n)z^{-n}=\sum_{n=-3}^{3}x(n)z^{-n}$$

$$X_2(z)=x(-3).z^{-(-3)}+x(-2).z^{-(-2)}+x(-1).z^{-(-1)}+x(0).z^{-(0)}$$

$$............+x(1)z^{-1}+x(2).z^{-2}+x(3)Z^{-3}$$

$$X_2(z)=0.z^{3}+1.z^{2}+3.z^{1}+5z^{0}+3.z^{-1}+1.z^{-2}+0.z^{-3}$$

$$X_2(z)=z^{2}+3z\ \ +5+3z^{-1}+z^{-2}$$

It is evident that the value of X(z) is infinite if z =0 or if z = ∞. So ROC is everywhere except z =0 and z = ∞.

(d) It is a causal sequence of infinite duration. The z-transform of the sequence is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X_4(z) = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$$

$$X_4(z) = 1 + z^{-1} + z^{-2} + \ldots\ldots$$

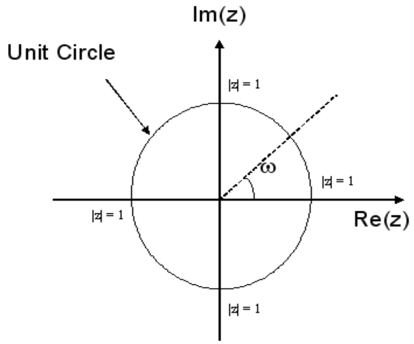
- This is a geometric series with a common ratio of Z^{-1}.
- Generally, geometric series with a common ratio of 'a' can be expressed as $\sum_{n=0}^{\infty} a^n = \dfrac{1}{1-a} \qquad |a| < 1.$
- Which is a closed form of power series with common ratio 'a' and the series converges if

- Thus, power series of $X_4(z)$ converges if $|z^{-1}| < 1$ *and equivalently if* $|z| > 1$.
- Thus, we may express X (z) in closed form provided that region of convergence (ROC) $|z| > 1$:

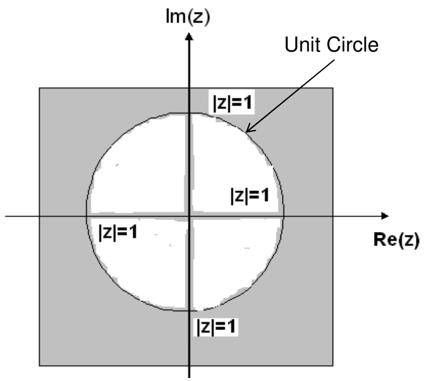
$$X(z) = 1 + z^{-1} + z^{-2} + \ldots\ldots$$

$$X(z) = \frac{1}{(1 - z^{-1})} = \frac{z}{z-1}$$

- In this case, the z-transform is valid everywhere outside a circle of unit radius (ie |z| =1) whose centre is at the origin. The exterior of the circle is the region of convergence.



The Unit Circle in the Complex z- Plane

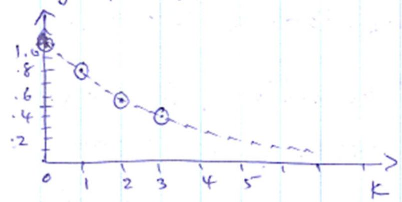


The ROC of sequence (d), the shaded area is ROC

z-Transform of a sampled exponential

$$y(k) = \begin{cases} 0 & \text{for } k < 0 \\ e^{-kT} & \text{for } k \geq 0 \end{cases}$$

say $T = 0.25$



k	0	1	2	3	--
y(k)	$e^{0} = 1$	$e^{-0.25} = 0.779$	$e^{-0.5} = 0.607$	$e^{-0.75} = 0.472$	--

$$Y(z) = \sum_{k=0}^{\alpha} y(k) z^{-k}$$

$$= \sum_{k=0}^{\alpha} e^{-kT} z^{-k}$$

$$= \sum_{k=0}^{\alpha} \underbrace{(e^{-T} z^{-1})}_{\alpha}{}^{k}$$

$$= 1 + \alpha + \alpha^2 + \ldots\ldots$$

$$\alpha Y(z) = \alpha + \alpha^2 + \alpha^3 + \ldots\ldots$$

$$(1 - \alpha) Y(z) = 1$$

$$Y(z) = \frac{1}{1-\alpha}$$

$$= \frac{1}{1 - e^{-T} z^{-1}}$$

$$= \frac{z}{z - e^{-T}}$$

Converging sequence for $|\alpha| < 1$

$$|e^{-T} z^{-1}| < 1$$

$$|z| > e^{-T}$$

ROC : Region of Convergence

	Sequence	z - transform
1	$\delta[n]$	1
2	$u[n]$	$\frac{z}{z-1}$
3	b^n	$\frac{z}{z-b}$
4	$b^{n-1}u[n-1]$	$\frac{1}{z-b}$
5	e^{an}	$\frac{z}{z-e^a}$
6	n	$\frac{z}{(z-1)^2}$
7	n^2	$\frac{z(z+1)}{(z-1)^3}$
8	$b^n n$	$\frac{bz}{(z-b)^2}$
9	$e^{an} n$	$\frac{z e^a}{(z-e^a)^2}$
10	$\sin(an)$	$\frac{\sin(a) z}{z^2 - 2\cos(a) z + 1}$
11	$b^n \sin(an)$	$\frac{\sin(a) b z}{z^2 - 2\cos(a) b z + b^2}$
12	$\cos(an)$	$\frac{z(z - \cos(a))}{z^2 - 2\cos(a) z + 1}$
13	$b^n \cos(an)$	$\frac{z(z - b\cos(a))}{z^2 - 2\cos(a) b z + b^2}$

The inverse z transform

- The inverse z-transform (IZT) allows us to recover the discrete time sequence, x(n), given its z-transform X(z).
- The inverse z-transform (IZT) is particularly useful in DSP work for example in finding the impulse response of digital filters.

$$x(n) = Z^{-1}[X(z)]$$

- If the z-transform appears as a power series of z as follows

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X_3(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + \ldots\ldots$$

Then the discrete time sequence x(0), x(1) …. Can be directly obtained by inspection.

- In practice, X (z) is often expressed as a ratio of two polynomials in z $^{-1}$ or equivalently in z,

$$X(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}$$

- In this form, the inverse z-transform, x (n), may be obtained using one of several methods including the following three:
  (a) Power series expansion method
  (b) Recursive method
  (c) Partial fraction expansion method

Power Series Method

- Given the z-transform X(z) of a causal sequence, it can be expanded into an infinite series in z $^{-1}$ or z by long division (sometimes called synthetic division):

$$X(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_N z^{-M}}$$

$$X(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + ...$$

- In this method, the numerator and denominator of X(z) are first expressed in either descending powers of z or ascending powers of z $^{-1}$ and the quotient is then obtained by long division.

Example

- Given the following z-transform of a causal LTI system, obtain its IZT (Inverse Z- Transform) by expanding it into a power series using long division:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + 0.3561 z^{-2}}$$

First, we expand X(z) into a power series with the numerator and denominator polynomials in ascending powers of z $^{-1}$ and then perform the usual long division.

$$\begin{array}{r|l}
 & 1 + 3z^{-1} + 3.6439 z^{-2} + 2.5756 z^{-3} + \ldots \\ \hline
1 - z^{-1} + 0.3561 z^{-2} & 1 + 2z^{-1} + z^{-2} \\
 & \underline{1 - z^{-1} + 0.3561 z^{-2}} \\
 & \quad 3z^{-1} + 0.6439 z^{-2} \\
 & \quad \underline{3z^{-1} - 3z^{-2} + 1.0683 z^{-3}} \\
 & \qquad 3.6439 z^{-2} - 1.0683 z^{-3} \\
 & \qquad \underline{3.6439 z^{-2} + 3.6439 z^{-3} + 1.2975927 z^{-4}} \\
 & \qquad\qquad 2.5756 z^{-3} - 1.2975927 z^{-4}
\end{array}$$

Alternatively, we may express Laplace Transform in +ve powers of z, in descending order and perform the long division.

Then, x(n)= x(0)=1, x(1)= 3, x(2)= 3.6439, x(4)= 2.5756, …..

Recursive Method

$$X(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_N z^{-M}}$$

$x(0) = a_0/b_0$

$x(1) = [a_1 - x(0)b_1]/b_0$

$x(2)=[a_2 - x(1)b_1 - x(0)b_2]/b_0$

Generally,

$$x(n) = \left[a_n - \sum_{i=1}^{n} x(n-i) b_i \right] / b_0, ...n = 1,2,3...$$

Example

Find the first four terms of the inverse z-transform, x (n), using the recursive approach. Assume that the z-transform, X(z), is the same as in example 4.5, that is:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + 0.3561 z^{-2}}$$

SOLUTION:

Comparing the coefficients of X(z) above with those of the general transform, we have,

$a_0=1$, $a_1=2$, $a_2=1$,

$b_0=1$, $b_1=-1$, $b_2=0.3561$,

N=M=2

By using general equation, we have,

$x(0) = a_0/ b_0 = 1/1 = 1$

$x(1) = [a_1 - x(0)\, b_1]/ b_0 = [2 - 1\times(-1)]/1 = 3$

$x(2) = [a_2 - x(1)\, b_1 - x(0)b_2]/b_0 = [1 - 3\times(-1) -1\times0.3561]/1$

$= 3.6439$

$x(3) = [a_3 - x(2)\, b_1 - x(1)b_2 + x(0)b_3]/ b_0$

$= 0 - x(2)\, b_1 - x(1)b_2$

$= [0 - 3.646439\times(-1) - 3\times0.3561]/1 = 2.5756$

$X(4) = [a_4 - x(3)\, b_1 - x(2)b_2 + x(1)\, b_3 + x(0)\, b_4]/ b_0$

$= [0 - 2.5756\times(-1) - 3.6439\times0.356 + 3\times0 + 1\times0]/1$

$= 2.5756 - 1.2972 + 0 + 0 = 1.2784$

Thus the first four values of the inverse z-transform are:

$x(0) =1$, $x(1) = 3$, $x(2) = 3.6439$, $x(3) = 2.5756$

It is seen that both the recursive and direct, long division methods lead to identical solutions.

Partial fraction expansion method

- In this method, the z-transform is first split into a sum of simple partial fractions. The inverse z-transform of each partial fraction is then obtained from z transform tables and then summed to give the overall inverse z-transform.
- In many practical cases, the z-transform is given as a ratio of polynomials in z or z^{-1}.

If the poles of X(z) are of first order, X(z) can be expanded as:

$$X(z)=\frac{a_0+a_1z^{-1}+a_2z^{-2}+......+a_Nz^{-N}}{b_0+b_1z^{-1}+b_2z^{-2}+......+b_Nz^{-M}}$$

$$X(z)=B+\frac{C_1}{1-b_1z^{-1}}+\frac{C_2}{1-b_2z^{-1}}+\cdots+\frac{C_M}{1-b_Mz^{-1}}$$

$$X(z)=B+\frac{C_1z}{z-b_1}+\frac{C_2z}{z-b_2}+\cdots+\frac{C_Mz}{z-b_M}$$

$$X(z)=B+\sum_{i=1}^{M}\frac{C_iz}{z-b_i}$$

where p_k are the poles of X(z) (assumed distinct), C_k are the partial fraction coefficients and

$B= a_N/b_N$

- Case 1: If N ≤ M ie the order of the numerator is less than that of the denominator in X(z) expression, then B_0 will be zero.
- Case 2: If N>M, then X(z) must be reduced first to make N ≤ M by long division.
- The coefficient, C_k, associated with the pole p_k may be obtained by multiplying both sides of the equation by (z - p_k) /z and then letting z = p_k.

$$C_i=\frac{X(z)}{z}(z-p_k)\Big|\, z=p_k$$

- Case 3: If X(z) contains one or more multiple-order poles (that is poles that are coincident) then extra terms are required in the partial fraction equation.
- For example, if X(z) contains an m^{th}-order pole at $z = p_k$ the partial fraction expansion must include terms of the form

$$\sum_{i=1}^{m}\frac{D_i}{(z-p_k)^i}$$

The coefficients, D_i, may be obtained from the relationship

$$D_i=\frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}[(z-p_k)^m X(z)]_{z=p_k}$$

Example

Find the inverse z-transform of the following transfer function H(z).

$$H(z)=\frac{(1-z^{-1})}{(1-z^{-1}-6z^{-2})}$$

Solution:
Since N < M, case 1.
First, H(z) is converted for positive power of z.

$$H(z)=\frac{1-z^{-1}}{1-z^{-1}-6z^{-2}}$$
$$=\frac{1-z^{-1}}{(1-3z^{-1})(1+2z^{-1})}$$

$$H(z)=\frac{A}{(1-3z^{-1})}+\frac{B}{(1+2z^{-1})}$$

To find A

$$A = H(z)(1-3z^{-1})\,|_{z^{-1}=1/3}$$

$$A = \frac{1-z^{-1}}{(1-3z^{-1})(1+2z^{-1})}(1-3z^{-1})\Bigg|_{z^{-1}=1/3} = \frac{1-z^{-1}}{(1+2z^{-1})}\Bigg|_{z^{-1}=1/3}$$

$$A = \frac{1-1/3}{(1+2\times 1/3)} = \frac{2/3}{5/3} = \frac{2}{5}$$

To find B

$$B = H(z)(1+2z^{-1})\,|_{z^{-1}=-1/2}$$

$$B = \frac{1-z^{-1}}{(1-3z^{-1})(1+2z^{-1})}(1+2z^{-1})\Bigg|_{z^{-1}=-1/2} = \frac{1-z^{-1}}{(1-3z^{-1})}\Bigg|_{z^{-1}=-1/2}$$

$$B = \frac{1-(-1/2)}{1-3\times(-1/2)} = \frac{1+1/2}{1+3/2} = \frac{3/2}{5/2} = 3/5$$

$$H(z) = \frac{2/5}{(1-3z^{-1})} + \frac{3/5}{(1+2z^{-1})}$$

By comparing to the z transform

$$x(n) = \alpha^n u(n)$$

$$X(z) = \frac{1}{1-\alpha z^{-1}}$$

The unit sample response of given system is:

$h(n) = (2/5)\ 3^n u(n) + 3/5\ (-2)^n u(n)$
$\quad = [(2/5)\ 3^n + 3/5\ (-2)^n]\ u(n)$

Example

Find the inverse z-transform of the following X(z):

$$X(z) = \frac{z^{-1}}{1-0.25z^{-1}-0.375z^{-2}}$$

Solution:
we get, N =1, M =2,
Thus, N<M ➔ Case 1: B_0 =0

- To change positive power of z by multiplying all terms with z^2 we get,

$$X(z) = \frac{z}{z^2-0.25z-0.375}$$

$$X(z) = \frac{z}{(z-0.75)(z+0.5)}$$

X(z) contains first-order poles at z= 0.75 and at z= -0.5.
Since N< M, the partial fraction expansion has the form,

$$\frac{X(z)}{z} = \frac{1}{(z-0.75)(z+0.5)} = \frac{C_1}{(z-0.75)} + \frac{C_2}{(z+0.5)}$$

To find C_1

$$C_1 = \frac{(z-0.75)X(z)}{z}\Bigg|_{z=0.75} = \frac{(z-0.75)1}{(z-0.75)(z+0.5)}\Bigg|_{z=0.75}$$

$$C_1 = \frac{1}{(z+0.5)}\Bigg|_{z=0.75} = \frac{1}{(0.75+0.5)} = 4/5$$

To find C_2

$$C_2 = \frac{(z+0.5)X(z)}{z}\Bigg|_{z=-0.5} = \frac{(z+0.5)1}{(z-0.75)(z+0.5)}\Bigg|_{z=-0.5}$$

$$C_2 = \frac{1}{(z-0.75)}\Bigg|_{z=-0.5} = \frac{1}{(-0.5-0.75)} = -4/5$$

Then

$$\frac{X(z)}{z} = \frac{C_1}{(z-0.75)} + \frac{C_2}{(z+0.5)} = \frac{4/5}{(z-0.75)} - \frac{4/5}{(z+0.5)}$$

$$X(z) = \frac{4/5.z}{(z-0.75)} - \frac{4/5.z}{(z+0.5)}$$

From the z-transform table $k\,a^n \Leftrightarrow k\,z/(z-a)$, therefore

$$Z^{-1}\left[\frac{4/5.z}{(z-0.75)}\right] = 4/5(0.75)^n \text{ and } Z^{-1}\left[\frac{-4/5.z}{(z+0.5)}\right] = -4/5(-0.5)^n$$

The desired inverse z-transform x(n) is then

$x(n) = 4/5\,[(0.75)^n - (-0.5)^n]$, $n>0$

Example

Find the discrete-time sequence x(n) with the following z-transform

$$X(z) = \frac{z^2}{(z-0.5)(z-1)^2}$$

Solution:

X(z) has a first-order pole at z = 0.5 and a second order pole at z =1. ➔ (Case 3)

The partial fraction expansion has the form,

$$\frac{X(z)}{z} = \frac{z}{(z-0.5)(z-1)^2} = \frac{C}{(z-0.5)} + \frac{D_1}{(z-1)} + \frac{D_2}{(z-1)^2}$$

$$C = \frac{(z-0.5)z}{(z-0.5)(z-1)^2}\Bigg|_{z=0.5} = \frac{z}{(z-1)^2}\Bigg|_{z=0.5} = \frac{0.5}{(0.5-1)^2} = 2$$

To obtain D_1 ,

$$D_1 = \frac{d}{dz}\left[\frac{(z-1)^2X(z)}{z}\right]_{z=1} = \frac{d}{dz}\left[\frac{(z-1)^2 z}{(z-0.5)(z-1)^2}\right]_{z=1} = \frac{d}{dz}\left[\frac{z}{z-0.5}\right]_{z=1}$$

$$D_1 = \frac{d}{dz}\left[\frac{z \to u}{z-0.5 \to v}\right]_{z=1} = \frac{vdu-udv}{v^2}$$

$$D_1 = \frac{(z-0.5).1-z.1}{(z-0.5)^2}\;_{z=1} = \frac{z-0.5-z}{(z-0.5)^2}\;_{z=1} = \frac{-0.5}{(z-0.5)^2}\;_{z=1}$$

To obtain D_2,

$$D_2 = \left.\frac{(z-1)^2 X(z)}{z}\right|_{z=1} = \left.\frac{z}{z-0.5}\right|_{z=1} = \frac{1}{1-0.5} = 2$$

Combining the results, X(z) becomes,

$$X(z) = \frac{2z}{(z-0.5)} - \frac{2z}{(z-1)} + \frac{2z}{(z-1)^2}$$

The inverse z-transform of each term on the right hand side is obtained from table 1 and summed to give x(n) as follows.

$x(n) = 2(0.5)^n - 2 + 2n = 2[(n-1) + (0.5)^n]$, $n \geq 0$

Example

Eg: $F(z) = \frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$

$\frac{F(z)}{z} = \frac{1-e^{-aT}}{(z-1)(z-e^{-aT})} = \frac{1}{(z-1)} - \frac{1}{(z-e^{-aT})}$

$F(z) = \frac{z}{z-1} - \frac{z}{z-e^{-aT}}$

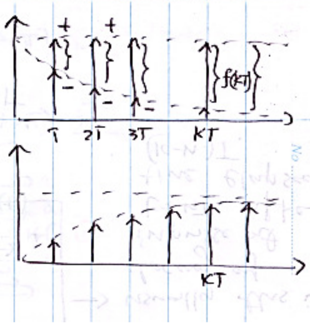
$f(kT) = 1 - e^{-akT}$

or

$f^*(t) = \sum_{k=0}^{\infty} f(kT)\,\delta(t-kT)$

value

position on time axis



Properties of the z-transform

(1) Linearity

If the sequences $x_1(n)$ and $x_2(n)$ have a z-transform $X_1(z)$ and $X_2(z)$, then the z-transform of their linear combination is:

$$ax_1(n) + bx_2(n) \Leftrightarrow aX_1(z) + bX_2(z)$$

and the ROC will *include* the intersection of R_x and R_y, that is $R_x \cap R_y$..

Shifting Property

Shifting a sequence (delaying or advancing) multiplies the z-transform by a power of z, that is to say, if x(n) has a z-transform X(z)

$$\mathbf{x(n-k) \Leftrightarrow z^{-k}X(z)}$$

Shifting does not change the region of convergence. Therefore, the z-transform of x(n) and x(n – k) have the same region of convergence.

Eg. $x(n-2) \Leftrightarrow z^{-2}X(z)$

z Transform of a shifted sequence

$$Z[y(k-l)] = \sum_{k=0}^{\infty} y(k-l) z^{-k}$$

introduce $i = k - l$ then as $k \to 0$ $i \to -l$, $k \to \infty$ $i \to \infty$

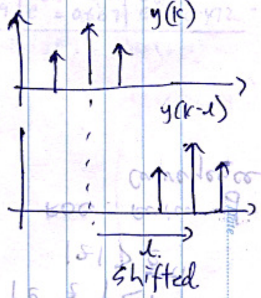
$$= \sum_{i=-l}^{\infty} y(i) z^{-i-l}$$

$$= \sum_{i=-l}^{0} 0 + \sum_{i=0}^{\infty} y(i) z^{-i} \cdot z^{-l}$$

$$= z^{-l} \sum_{i=0}^{\infty} y(i) z^{-i}$$

$$Z[y(k-l)] = z^{-l} Y(z)$$

time delay operator



Properties: Time Reversal

If x(n) has a z-transform X(z) with a region of convergence R_x that is the annulus $\alpha < |z| < \beta$, the z-transform of the time-reversal sequence x (-n) is

$$\mathbf{x(-n) \Leftrightarrow X(z^{-1})}$$

and has a region of convergence $1/\beta < |z| < 1/\alpha$, which is denoted by $1/R_x$.

Multiplication by an Exponential

If a sequence x(n) is multiplied by a complex exponential α^n,

$$\mathbf{\alpha^n x(n) \Leftrightarrow X(\alpha^{-1} z)}$$

This corresponds to a scaling of the z-plane. If the region of convergence of X(z) is $r_- < |z| < r_+$, which will be denoted by R_x, the ROC of $X(\alpha^{-1}z)$ is $|\alpha| r_- < |z| < |\alpha| r_+$, which is denoted by $|\alpha| R_x$. As a special case, note that if x(n) is multiplied by a complex exponential, $e^{jn\omega 0}$,

$$\mathbf{e^{jn\omega 0} x(n) \Leftrightarrow X(e^{-j\omega 0} z)}$$

which corresponds to a rotation of the z-plane.

Properties: Convolution Theorem

Perhaps the most important z-transform property is the convolution theorem, which states that convolution in the time domain is mapped into multiplication in the frequency domain, that is,

$$y(n) = x(n) * h(n) \leftrightarrow Y(z) = X(z)H(z)$$

The region of convergence of Y(z) includes the interaction of R_x and R_y, R_w contains $R_x \cap R_y$ However, the region of convergence of Y(z) may be larger, if there is a pole-zero cancellation in the product X(z)H(z).

Real Convolution Theorem

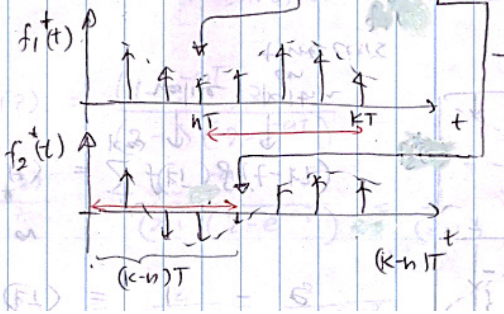
$f_1(t) \longrightarrow F_1(z)$; $f_1(t) = 0$

$f_2(t) \longrightarrow F_2(z)$; $f_2(t) = 0$ $\Big\}$ $\forall\ t < 0$

convolution sum

$$f_1(kT) * f_2(kT) = \sum_{n=0}^{k} f_1(nT)\, f_2(kT - nT)$$

For a given n - convolution length

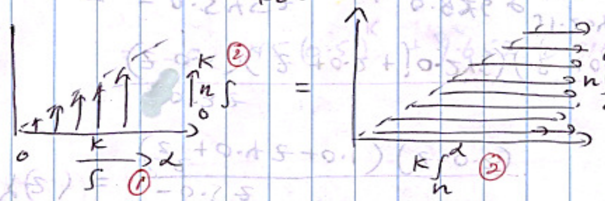


usually this is delay of impulse at $t = nT$ after time elapsed $(k-n)T$.

$$= f_1(0)\, f_2(kT) + f_1(T)\, f_2[(k-1)T] + \cdots$$

$$Z\left[f_1(kT) * f_2(kT)\right] = \sum_{k=0}^{\infty} \sum_{n=0}^{k} f_1(nT)\, f_2(kT - nT)\, z^{-k}$$

Generalized for any n



$$= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} f_1(nT)\, f_2((k-n)T)\, z^{-k}$$

Introduce $m = k - n$ then as $k \to n$, $m \to 0$; $k \to \infty$, $m \to \infty$

$$\therefore = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_1(nT)\, f_2(mT)\, z^{-m-n}$$

$$= \sum_{n=0}^{\infty} f_1(nT) z^{-n} \sum_{m=0}^{\infty} f_2(mT)\, z^{-m}$$

$$= F_1(z) \cdot F_2(z) \quad //$$

Example

Consider the two sequences

$x(n) = \alpha^n u(n)$ and

$h(n) = \delta(n) - \alpha\delta(n - 1)$

If y(n) is convolution of x(n) and h(n), find the sequence y(n)

Solution:

The z-transform of x(n) is

$X(z) = 1/(1 - \alpha z^{-1})$ $\quad |z| > |\alpha|$ and

the z-transform of h(n) is

$H(z) = 1 - \alpha z^{-1}$ $\quad |z| > 0$

- However, the z-transform of the convolution of x(n) with h(n) is $Y(z) = X(z)H(z) = 1/(1 - \alpha z^{-1})(1 - \alpha z^{-1}) = 1$
- which, due to a *pole-zero* cancellation, has a region of convergence, that is the entire z-plane. By taking Inverse z-transform, $y(n) = \delta(n)$

Properties: Derivative

If X(z) is the z-transform of x(n), the z-transform of nx(n) is

$$nx(n) \leftrightarrow -z\frac{dX(z)}{dz}$$

Repeated application of this property allows for the evaluation of the z-transform of $n^k x(n)$ for any integer k.

Table 2: Properties of z- transform

Property	Sequence	z-Transform	RoC
Linearity	ax(n) + by(n)	aX(z) + bY(z)	Contains $R_x \cap R_y$
Shift	$x(n - n_0)$	$z^{-n_0} X(z)$	R_x
Time Reversal	x(-n)	$X(z^{-1})$	$1/R_x$
Exponential	$\alpha^n x(n)$	$X(\alpha^{-1} z)$	αR_x
Convolution	$x(n) * y(n)$	X(z) Y(z)	Contains $R_x \cap R_y$
Conjugation	x*(n)	X*(z*)	R_x
Derivative	nx(n)	$-z\frac{dX(z)}{dz}$	R_x

Initial Value Theorem

A property that may be used to find the initial value of a causal sequence from its z-transform is the initial value theorem.

If x(n) is causal, ie x(n)=0 for n<0], then

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

and

$$x(0) = \lim_{z \to \infty} X(z)$$

Example

Let x(n) be a left-sided sequence that is equal to zero for n > 0. If $X(z) = (3z^{-1} + 2z^{-2})/(3 - z^{-1} + z^{-2})$, find x(0).

Solution:

For a left-sided sequence that is zero for n>0, the z-transform is $X(z) = x(0) + x(-1)z + x(-2)z^2 + \ldots\ldots\ldots\ldots$

Therefore, it follows that

$$x(0) = \lim_{|z| \to 0} X(z)$$

For the given z-transform, we see that

$$x(0) = \lim_{z \to 0} X(z) = \lim_{z \to 0} \frac{3z^{-1} + 2z^{-2}}{3 - z^{-1} + z^{-2}} \times \frac{z^2}{z^2} = \lim_{z \to 0} \frac{3z + 2}{3z^2 - z + 1} = 2$$

Example

Generalize the initial value theorem to find the value of a causal sequence x(n) at n = 1, and find x(1) when $X(z) = (2 + 6z^{-1})/(4 - 2z^{-2} + 13z^{-3})$.

Solution:

If x(n) is causal,

$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots$

Therefore, note that if we subtract x(0) from X(z),

$X(z) - x(0) = x(1)z^{-1} + x(2)z^{-2} + \ldots$

By multiplying both sides of this equation by z, we have,

$z[X(z) - x(0)] = x(1) + x(2)z^{-1} + \ldots$

If we let z → ∞, we obtain the value for x(1), therefore,

$x(1) = \lim_{|z| \to \infty} \{z[X(z) - x(0)]\}$

Example: For the z-transform $X(z) = \dfrac{2+6z^{-1}}{4-2z^{-2}+13z^{-3}}$

$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{2+6z^{-1}}{4-2z^{-2}+13z^{-3}} = \frac{1}{2}$$

Then, X(z) – x(0) = X(z) - ½ is:

$$X(z) - \frac{1}{2} = \frac{2+6z^{-1}}{4-2z^{-2}+13z^{-3}} - \frac{1}{2}$$

$$= \frac{4+12z^{-1}-4+2z^{-2}-13z^{-3}}{2(4-2z^{-2}+13z^{-3})} = \frac{12z^{-1}+2z^{-2}-13z^{-3}}{2(4-2z^{-2}+13z^{-3})}$$

$$= \frac{6z^{-1}+z^{-2}-13/2z^{-3}}{(4-2z^{-2}+13z^{-3})}$$

Then,

$$z[X(z)-x(0)] = z \times \frac{6z^{-1}+z^{-2}+13/2z^{-3}}{4-2z^{-2}+13z^{-3}}$$

$$z[X(z)-x(0)] = \frac{6+z^{-1}+13/2z^{-2}}{4-2z^{-2}+13z^{-3}}$$

Consequently,

$$x(1) = \lim_{z\to\infty} \{z[X(z) - x(0)]\}$$

$$x(1) = \lim_{z\to\infty} \frac{6+z^{-1}+13/2z^{-2}}{4-2z^{-2}+13z^{-3}} = \frac{3}{2}$$

The One-Sided z-Transform - shifting

- The *one-sided*, or *unilateral*, z-transform is defined by:

$$X_1(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

- The primary use of the one-sided z-transform is to solve linear constant coefficient difference equations that have initial conditions.

- Most of the properties of the one-sided z-transform are the same as those for the two-sided z-transform. One that is different, however, is the shift property.
- Specifically, if x(n) has a one-sided z-transform X(z), the one-sided z-transform of x(n – k) where k > 0 is:

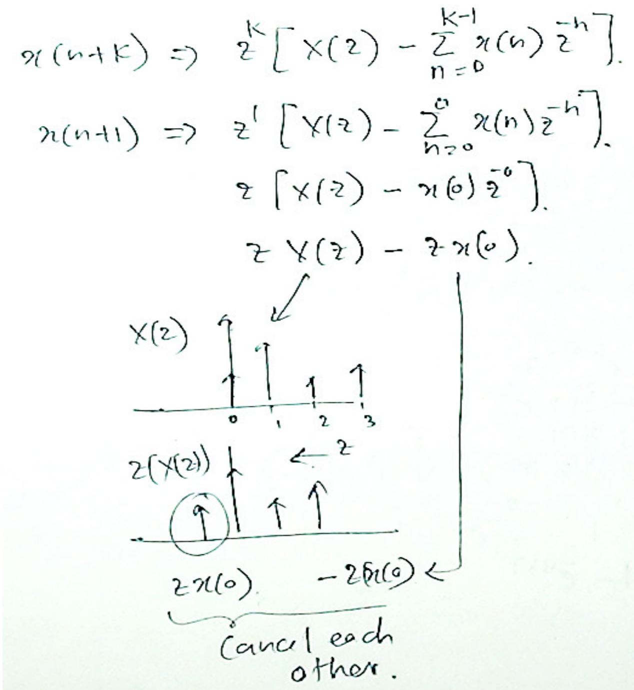
$$x(n-k) \Rightarrow z^{-k}\left[X_1(z) + \sum_{n=1}^{k} x(-n) z^{n}\right]$$

- Then, one-sided z-transform of x(n – 1) and x(n-2) are:

$$x(n-1) \Rightarrow z^{-1}X_1(z) + x(-1)$$

$$x(n-2) \Rightarrow z^{-2}X_1(z) + z^{-1}x(-1) + x(-2)$$

Left shift



Application of z Transforms in Discrete Systems Solving Difference Equations

Example

Find the solution of the linear constant coefficient difference equation

y(n)=0.25 y(n–2) + x(n) for x(n)=δ(n–1) with y(-1)=y(-2)=1.

Solution:

If the one-sided z-transform of y(n) is Y(z), the one-sided z-transform of y(n–2) is

$$\sum_{n=0}^{\infty} y(n-2)z^{-n} = z^{-2}Y(z) + y(-1)z^{-1} + y(-2)$$

Therefore, taking the z-transform of both sides of the difference equation,

- y(n) = 0.25y(n – 2) + x(n), we have
- $Y(z) = 0.25[y(-2) + y(-1)z^{-1} + z^{-2}Y(z)] + X(z)$

For x(n) = δ(n – 1)

- $X(z) = z^{-1} \times Z[\delta(n)] = z^{-1} \times 1 = z^{-1}$

Substituting for y(-1) and y(-2), and solving for $Y_1(z)$, we have

$Y(z) = 0.25[1 + z^{-1} + z^{-2}Y(z)] + z^{-1}$

$Y(z) = ¼ + ¼ z^{-1} + ¼ z^{-2}Y(z) + z^{-1}$

$Y(z) - ¼ z^{-2}Y_1(z) = ¼ + 5/4 z^{-1}$

$Y(z)[1 - ¼ z^{-2}] = ¼ [(1 + 5z^{-1})]$

$Y(z) = ¼ (1 + 5z^{-1})]/ [1 - ¼ z^{-2}]$

- $Y_1(z) = ¼ (1 + 5z^{-1})]/ [1 - ¼ z^{-2}]$

$$Y(z) = \frac{\frac{1}{4}(1+5z^{-1})}{1-\frac{1}{4}z^{-2}} = \frac{\frac{1}{4}(1+5z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}$$

$$Y(z) = \frac{A}{(1-\frac{1}{2}z^{-1})} + \frac{B}{(1+\frac{1}{2}z^{-1})}$$

$$A = H(z)\left(1 - 1/2z^{-1}\right)\Big|_{z^{-1}=2}$$

$$A = \frac{1/4(1+5z^{-1})}{(1-1/2z^{-1})(1+1/2z^{-1})}\left(1-1/2z^{-1}\right)\Bigg|_{z^{-1}=2} = \frac{1/4(1+5z^{-1})}{(1+1/2z^{-1})}\Bigg|_{z^{-1}=2}$$

$$A = \frac{1/4(1+5\times 2)}{1+1/2\times 2} = \frac{11/4}{2} = \frac{11}{8}$$

$$B = H(z)(1+1/2z^{-1})\Big|_{z^{-1}=-2} = \frac{1/4(1+5z^{-1})}{(1-1/2z^{-1})}\Bigg|_{z^{-1}=-2} = \frac{1/4[1+5(-2)]}{1-1/2\times(-2)}$$

$$B = \frac{-9/4}{2} = -9/8$$

Then $$Y_1(z) = \frac{11/8}{(1-1/2z^{-1})} - \frac{9/8}{(1+1/2z^{-1})}$$

And y(n) = $[(11/8)(1/2)^n - (9/8)(-1/2)^n]u(n)$

Example

A discretized system is given by

$$y(k+2) + 0.4\,y(k+1) + 0.1\,y(k) = -0.3\;0.5^k$$

recall that $Z[a^k] = \frac{z}{z-a}$

Take z-trans fun

$$z^2 Y(z) + 0.4\,z\,Y(z) + 0.1\,Y(z) = -0.3\,\frac{z}{z-0.5}$$

$$Y(z) = \frac{-0.3z}{(z^2+0.4z+0.1)(z-0.5)}$$

$$= \frac{-0.3z}{(z-0.5)(z+0.2+j0.245)(z+0.2-j0.245)}$$

Assignment #3: Solve for y(k)

Car Loan Problem

Car loan Problem

Calculate monthly installment

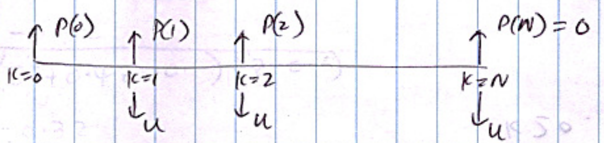
$P(k)$ amount owed at the beginning of k^{th} period (month)

$$P(k+1) = (1+R)\,P(k) - U$$

$P(k+1)$: amount owe at the end of period (of one month)

R: monthly interest %

U: monthly installment



take z tr of both sides

$$\sum_{k=0}^{\infty} P(k+1)\,z^{-k} = \sum_{k=0}^{\infty} (1+R)\,P(k)\,z^{-k} - \sum_{k=0}^{\infty} U\,z^{-k}$$

Assignment #4 : Determine U